JOURNAL OF FUNCTIONAL ANALYSIS 31, 187–194 (1979)

A Class of Symmetric and a Class of Wiener Group Algebras

JEAN LUDWIG

Universität Bielefeld, Bielefeld 4800, Federal Republic of Germany

Communicated by P. Malliavin

Received January 1977

It is shown that connected groups of polynomial growth and compact extensions of nilpotent group have symmetric group algebras and that the group algebras of discrete solvable groups have the Wiener property.

I. SYMMETRY

A Banach algebra A with involution $*$ is symmetric if any element of the form x^*x, $x \in A$, has positive spectrum. This is equivalent to the fact that hermitian elements of A have real spectrum.

In [12] it is shown that a Banach $*$ algebra A is symmetric if and only if each proper modular left Ideal I is contained in the kernel of a positive hermitian continuous functional $p \neq 0$.

Let G be a locally compact group and let dx be the left Haar measure of G; let A be the group algebra $L^1(G, dx) = L^1(G)$. Symmetry in group algebras has been investigated among others in [11], where many examples of symmetric group algebras can be found.

In [7] J. W. Jenkins proved the following:

Let G be an exponentially bounded group. Take f in $L^1(G)$ and let $\sigma_p(f)$ denote the spectrum of f, when considered as a bounded convolution operator on $L^p(G)(1 \leqslant p < \infty)$. Then for any $f \in L_0^1(G)$, the subalgebra of rapidly decreasing functions: $\sigma_p(f) = \sigma_2(f)$ for all $1 \leqslant p < \infty$.

But in [3] an example of a discrete locally finite (hence exponentially bounded) group is given, whose l^1 algebra is not symmetric.

Thus the result of Jenkins cannot be generalized to all f in $L^1(G)$. We are able to prove the symmetry of $L^1(G)$, if G is exponentially bounded and connected.

First we give a characterization of symmetric group algebras in terms of the group G.

Suppose that the group algebra $L^1(G)$ of G is symmetric. Let I be a modular proper left ideal of $L^1(G)$. Then there exists a positive functional p on $L^1(G)$, $p \neq 0$, such that $\langle p, I\rangle = 0$, p defines a positive hermitian sesquilinear bounded

0022-1236/79/020187-08$02.00/0

functional B_p on $L^1(G) \times L^1(G)$: $B_p(f, g) = p(g^* f)(f, g \in L^1(G))$ such that $B_p(I, L^1(G)) = 0$.

Denote by $\mathbf{S}$ the bounded sesquilinear positive hermitian functionals on $L^1(G)$. The group G acts on the left and on the right on $L^1(G)$ by ${}_tf(s) = f(t^{-1}s)$ and $f_t(s) = f(st)\,\Delta(t)$ for all $s, t \in G, f \in L^1(G)$ where Δ denotes the Haar modular function of G. Thus G acts on $\mathbf{S}$ too on the left and on the right by ${}_sB(f, g) := B({}_sf, {}_sg)$ and $B_s(f, g) := B(f_s, g_s)\,\Delta^2(s)$ for all $s \in G$, $f, g \in L^1(G)$. Let F be a subspace of $L^1(G)$ and H a subgroup of G; let $\mathbf{S}_F{}^H \subset \mathbf{S}$ be defined by: $\mathbf{S}_F{}^H = \{B \in \mathbf{S} \mid {}_hB = B$ for all $h \in H$ and $B(F, L^1(G)) = 0\}$. The element B_p of $\mathbf{S}$ is obviously contained in $\mathbf{S}_I{}^G$. Thus $\mathbf{S}_I{}^G \neq (0)$.

Suppose on the other hand that for all modular left ideals of $L^1(G)$ $\mathbf{S}_I{}^G \neq (0)$. Let α be the modular right unit of I. Then for all $B \in \mathbf{S}_I{}^G - (0)$: $B(\alpha, \alpha) > 0$, because otherwise:

$$| B(f, g)| = | B(f * \alpha, g)| = | B(\alpha, f^* * g)| \leqslant B(\alpha, \alpha)^{1/2}\, B(f^* * g, f^* * g)^{1/2} = 0$$

for all $f, g \in L^1(G)$.

The linear functional p_B defined on $L^1(G)$ by $p_B(f) := B(f * \alpha, \alpha) = B(f, \alpha)$ is positive and not zero and $p_B(I) = 0$. Thus we have proved:

(1) *The group algebra $L^1(G)$ of a locally compact group G is symmetric if and only if for all modular left ideals I, $\mathbf{S}_I \neq 0$ implies $\mathbf{S}_I{}^G \neq 0$.*

Our main tools to prove this fixpoint property for certain groups are the approximately constant functions p which exists in the L^1 algebra of exponentially bounded groups and which had been introduced by J. W. Jenkins in [7]. A locally compact group is said to be exponentially bounded if for each compact neighbourhood U of the identity in G lim sup $| U^n |^{1/n} = 1$, where $| \cdot |$ denotes the left Haar measure of a set and $U^n = \{u_1 u_2 \cdots u_n \mid u_i \in U,\ 1 \leqslant i \leqslant n\}$. If G is exponentially bounded, then for any compact subset U of G and any $\epsilon > 0$, there is a $\rho \in L^1(G)$ such that $\rho(t) > 0$ and $| \rho(st) - \rho(t)| \leqslant \epsilon\rho(t)$ for all $s \in U$ for all t in the group $\langle U \rangle$ generated by U. Take any compact neighborhood $V = V^{-1}$ of e in G containing U. Define ρ by $\rho(t) = (1 + \epsilon)^{-k}$ if $t \in V^{k+1} \backslash V^k$. (See Lemma 1 of [8]). We say that a closed normal subgroup N of a group G is an *s.i. subgroup* if each $n \in N$ is contained in a compact, G invariant neighborhood of the identity element e; an s.i. subgroup of a group G has polynomial growth for it is an $[FC]^-$ group (see [5].)

Now we can state our main lemma.

LEMMA 1. *Let G be a locally compact group, H and N normal closed subgroups of G such that $N \subset H$ and such that H/N is an s.i. subgroup of G/N. Let I be a closed modular left ideal in $L^1(G)$ with modular right unit α.*
Then:

$$\mathbf{S}_I{}^N \neq (0) \textit{ implies } \mathbf{S}_I{}^H \neq (0)$$

Proof. Let $D \in \mathbf{S}$ and $f, g \in L^1(G)$. The following inequalities hold:

$$D(f * g, f * g) \leqslant |f|_1 \int_G |f(x)|\, {}_xD(g, g)\, dx \tag{1}$$

$$D(f * g, f * g) \leqslant |g|_1 \int_G |g(x)|\, D_{x^{-1}}(f, f)\, dx \tag{2}$$

because:

$$\begin{aligned} D(f * g, f * g) &= \int_G f(x)\, D({}_xg, f * g)\, dx \\ &\leqslant \int_G |f(x)|^{1/2}\, |f(x)|^{1/2}\, |D({}_xg, f * g)|\, dx \\ &\leqslant |f|_1^{1/2} \cdot \left(\int |f(x)|\, D({}_xg, {}_xg)\, dx\right)^{1/2} \cdot D(f * g, f * g)^{1/2} \end{aligned}$$

This proves (1); the proof of (2) is similar.

The group G/N acts on the left on $\mathbf{S}_I^N$ as

$${}_{nx}B(f, g) = {}_nB({}_xf, {}_xg) = B({}_xf, {}_xg) = {}_xB(f, g)$$

for all $B \in \mathbf{S}_I^N$, for all $f, g \in L^1(G)$ for all $n \in N$, for all $x \in G$.

We show that for any compact set $K \subset H/N$ and for any $\epsilon > 0$ there exists $\tilde{B} \in \mathbf{S}_I^N$, such that $\tilde{B}(\alpha, \alpha) \geqslant \frac{1}{2}$

$$|B| = (\sup_{\substack{|f|=1 \\ |g|=1}} B(f, g)) \leqslant 1 \qquad \text{and such that} \quad |{}_kB(f, f) - B(f, f)| \leqslant \epsilon\, |f|_1^2$$

for all $f \in L^1(G)$, for all $k \in K$, if $S_I^N \neq (0)$. Suppose that $S_I^N \neq (0)$. Let K be a compact subset of H/N and let $\epsilon > 0$. K is contained in a G/N invariant compact neighborhood $U = U^{-1}$ of N in H/N as H/N is an s.i. subgroup of G/N. The group V generated by U is G/N invariant and open in H/N. There exists $\rho \in L^1(V)$ such that $\rho(v) > 0$ and $|\rho(kv) - \rho(v)| \leqslant \epsilon\rho(v)$ for all $v \in V$, $k \in K$, as V is exponentially bounded.

We have for all $E \in \mathbf{S}_I$:

$$E(f * \alpha, g) = E(f, g); \qquad f, g \in L^1(G).$$

Thus (1) implies:

$$\begin{aligned} E(f, f) = E(f * \alpha, f * \alpha) &\leqslant |f|_1 \int_G |f(x)|\, {}_xE(\alpha, \alpha)\, dx \\ &\leqslant |f|_1^2\, (\sup_{x \in G} {}_xE(\alpha, \alpha)) \end{aligned} \tag{3}$$

Take $0 \neq B \in \mathbf{S}_I^N$.

Define $B' \in \mathbf{S}_I^N$ by:

$$B'(f, g) := \int_V \rho(v)\, {}_{v^{-1}}B(f, g)\, dv, \qquad f, g \in L^1(G).$$

Choose $y \in G$ such that: ${}_yB'(\alpha, \alpha) \geqslant \frac{1}{2}b$ where

$$b = (\sup_{x \in G} {}_xB'(\alpha, \alpha))$$

Define $\tilde{B} \in \mathbf{S}_I^N$; by $\tilde{B} = 1/b \cdot {}_yB'$;
Then: (a) $|\tilde{B}| \leqslant 1$ because:

$$\begin{aligned}
|\tilde{B}(f, g)| &\leqslant (\tilde{B}(f, f))^{1/2} \cdot \tilde{B}(g, g)^{1/2} \\
&\leqslant |f|_1 \cdot |g|_1 \cdot \sup_{x \in G} {}_x\tilde{B}(\alpha, \alpha) \\
&= |f|_1 \cdot |g|_1 \cdot \sup_{x \in G} \left(\frac{1}{b} \cdot \int_V \rho(v)\, B({}_{v^{-1}yx}\alpha, {}_{v^{-1}yx}a)\, dv\right) \\
&\leqslant |f|_1\, |g|_1
\end{aligned}$$

(b) $\tilde{B}(\alpha, \alpha) \geqslant \frac{1}{2}$ because:

$$\tilde{B}(\alpha, \alpha) = \frac{1}{b}\, {}_yB'(\alpha, \alpha) \geqslant \frac{1}{2}\frac{b}{b} = \frac{1}{2}$$

(c) $|{}_u\tilde{B}(f, f) - \tilde{B}(f, f)| \leqslant \epsilon\, |f|_1^2$ for all $f \in L^1(G)$, for all $u \in U$ because:

$$\begin{aligned}
|{}_u\tilde{B}(f, f) - \tilde{B}(f, f)| &= \frac{1}{b}\left| \int_V \rho(v)({}_{v^{-1}yu}B(\alpha, \alpha) - {}_{v^{-1}y}B(\alpha, \alpha))\, dv \right| \\
&\leqslant \frac{1}{b} \int_V |\rho(yuy^{-1}v) - \rho(v)|\, {}_{v^{-1}y}B(\alpha, \alpha)\, dv \\
&\leqslant \frac{\epsilon}{b} \int \rho(v)\, {}_{v^{-1}y}B(\alpha, \alpha)\, dv \\
&= \epsilon \tilde{B}(f, f), \qquad (\text{as } yuy^{-1} \in U) \\
&\leqslant \epsilon\, |f|_1^2
\end{aligned}$$

For K compact in H/N and for $\epsilon > 0$ let

$$\begin{aligned}
A_{K,\epsilon} = \{ B \in \mathbf{S}_I^H \mid |B| \leqslant 1,\ B(\alpha, \alpha) \geqslant \tfrac{1}{2}, \\
|{}_kB(f, f) - B(f, f)| \leqslant \epsilon\, |f|_1^2 \text{ for all } f \in L^1(G), \text{ for all } k \in K\}
\end{aligned}$$

$A_{K,\epsilon}$ is weak $*$ closed in ${}^1\mathbf{S}_I^N = \mathbf{S}_I^N \cap \{ D \in \mathbf{S} \mid |D| \leqslant 1\}$. We have just seen that the intersection of finitely many sets $A_{K,\epsilon}$ is non empty, the compactness of ${}^1\mathbf{S}_I^N$ in the weak $*$ topology implies: $\bigcap_{K,\epsilon} A_{K,\epsilon} \neq \phi$ (K compact, $\epsilon > 0$).

If B is in this intersection we have:

$${}_hB(f, f) = B(f, f) \qquad \text{for all } h \in H, \quad \text{for all } f \in L^1(G)$$

But by polarization we get ${}_hB = B$ for all $h \in H$, that means:

$$\mathbf{S}_I^H \neq (0).$$

An easy application of the lemma shows that

(a) *Compact extensions of nilpotent groups and*

(b) *connected groups of polynomial growth have symmetric group algebras*; because

(a) the centre of the nilpotent normal subgroup N of the compact extension G of N is an s.i. subgroup of G.

(b) If G is connected and of polynomial growth, then there exists a sequence of closed normal subgroup $\{G_i\}_{i=0,1,2,\ldots,n}$ such that $\{e\} = G_0 \subset G_1 c \cdots c G_n = G$ and such that $G_{i+1}/_{G_i}$ is an s.i. subgroup of G/G_i, $i = 0,\ldots, n-1$ (see [4]).

Remark. D. Poguntke has proved symmetry for connected nilpotent Liegroups in [13] and A. Hulanicki for discrete nilpotent groups in: [6]

II. Wiener property

A locally compact group has the Wiener property or $[W]$ property if every closed twosided ideal in the Banach $*$ algebra $L^1(G)$ is contained in the kernel of a non degenerate continuous $*$ representation of $L^1(G)$ on a Hilbertspace. Compact and abelian groups have the $[W]$ property of course. In [10] it is shown that connected nilpotent Liegroups and semi-direct products of abelian groups have the $[W]$ property too.

A group G is said to be weakly Wiener if every closed proper twosided ideal is contained in some primitive ideal. A group with symmetric group algebra, which is weakly Wiener has always the $[W]$ property. In ([10], (10)) it is shown that a Lie group G with polynomial growth is weakly Wiener. One can easily generalize this result:

Lemma 2. *Any locally compact group G with polynomial growth has the weak Wiener property.*

Proof. If G is of polynomial growth, and if $f = f^*$ is a continuous function with compact support, then the compactly supported real or complex valued functions which are n times differentiable (n depending on the support of f) and vanishing at 0, operate on f. (See [2]).

Proceeding as in the proof of Théorème 1 of [2], we can find for f in $L^1(G)$ and for the function φ defined in part b of the proof of Théorème 1 a function ψ such that $\psi\{f\} * \varphi\{f\} = (\psi \cdot \varphi)\{f\} = \varphi\{f\}$ in the notations of [2]. Thus one can prove that there exists in $L^1(G)$ an approximate identity $\{e_\lambda\}_{\lambda\in\Lambda}$ such that for each $\lambda \in \Lambda$ there exists $b_\lambda \in L^1(G)$ with $b_\lambda * e_\lambda = e_\lambda$. Now, proceed as in the proof of (10) in [9]. Q.E.D.

As the groups in lemma 1 have polynomial growth, we have:

Compact extensions of nilpotent groups and connected groups of polynomial growth have the Wiener property.

We can reformulate the Wiener property condition as we did in part I for the symmetry condition:

A group G has the Wiener property if and only if for every twosided ideal I in $L^1(G)$:

$$\mathbf{S}_I \neq 0 \Rightarrow \mathbf{S}_I^G \neq (0)$$

Now we are going to prove the $[W]$ property for discrete solvable groups.

LEMMA 3. *Let G be a locally compact group, H and N closed subgroups of G, such that $N \subset H$, N is normal in H and such that H/N is exponentially bounded. Let I be a closed twosided ideal in $L^1(G)$ with a right modular unit α. If $\mathbf{S}_I^N \neq (0)$ then $\mathbf{S}_I^N \neq (0)$.*

Proof. Let $B \in \mathbf{S}_I$, $f, g \in L^1(G)$, $t \in G$. As α is a modular right unit for I and as I is a closed twosided ideal we have $f * \alpha * g - f * g \in I$, and thus $\alpha * g - g \in I$ and $f_t * \alpha - f * \alpha_t \in I$. Thus: $B(f_t, g) = B(f_t * \alpha, g) = B(f * \alpha_t, g)$. And $B(f, g) = B(\alpha * f, g)$. Suppose that $\mathbf{S}_I^N \neq (0)$. First we prove that for all K compact in H/N and for all $\epsilon > 0$ there exists $T \in \mathbf{S}_I^N$ such that $T(\alpha, \alpha) \geqslant \frac{1}{2}$, $|T| \leqslant 1$ and $|{}_kT(f, f) - T(f, f)| < \epsilon\, |f|_1^2$ for all $k \in K$. Take $B \in \mathbf{S}_I^N$ with $B(\alpha, \alpha) > 0$; let K be a compact subset of H/N, let $U = U^{-1}$ be a compact neighborhood of K in H/N containing the unit element of H/N and take $\epsilon > 0$.

As H/N is exponentially bounded, there exists $\rho \in L^1(H/N)$, such that $\rho(t) > 0$ and $|\rho(ut) - \rho(t)| < \epsilon\rho(t)$ for all $u \in U$, for all t in the group V generated by U.

Define $B' \in \mathbf{S}_I^N$ by:

$$B'(f, g) = \int_V \rho(v)\, {}_{\bar v^1}B(f, g)\, dv \qquad \text{for} \quad f, g \in L^1(G)$$

Let $b = \sup_{y\in G} (B'_y(\alpha, \alpha))$ and choose $x_0 \in G$ such that

$$B'_{x_0}(\alpha, \alpha) \geqslant \tfrac{1}{2}b.$$

Define $T \in \mathbf{S}_I^N$ by:

$$T := \frac{1}{b} \cdot B'_{x_0}$$

We have: $| T | \leqslant 1$ because:

$$T(f,f) = T(\alpha * f, \alpha * f) \leqslant |f|_1 \cdot \int_G |f(x)| \, B'_{\bar{x}^1}(\alpha, \alpha) \, dx$$

$$\leqslant |f|_1^2 \sup_{x \in G} (B'_{\bar{x}^1}(\alpha, \alpha))$$

$$= |f|_1^2 \cdot \frac{1}{b} \left(\sup_{x \in G} \left(\int_V \rho(v)({}_{v^1}B_{x_0})_{\bar{x}^1}(\alpha, \alpha) \right) \right) dv$$

$$\leqslant |f|_1^2$$

$T(\alpha, \alpha) \geqslant \frac{1}{2}$ because $T(\alpha, \alpha) = 1/b \; B'(\alpha, \alpha)_{x_0} \geqslant b/2 \cdot 1/b = \frac{1}{2} \cdot$
$| {}_uT(f, f) - T(f,f)| < \epsilon \, |f|_1$ *for all* $u \in U$, *for all* $f \in L^1(G)$ because:

$$| {}_uT(f,f) - T(f,f)| = \frac{1}{b} \left| \int_V \rho(v)({}_{\bar{v}^1 u}B(f,f) - {}_{\bar{v}^1}B(f,f)) \, dv \right|$$

$$\leqslant \frac{1}{b} \int_V | \rho(uv) - \rho(v)| \; {}_{\bar{v}^1}B(f,f) \, dv$$

$$\leqslant \epsilon T(f,f) \leqslant \epsilon \, |f|_1^2$$

For K compact in H/N, for $\epsilon > 0$ let

$$A_{K,\epsilon} = \{B \in \mathbf{S}_I^N \mid B(\alpha, \alpha) \geqslant \tfrac{1}{2}; \, | B | \leqslant 1; \, | {}_kB(f,f) - B(f,f) \leqslant \epsilon \, |f|_1^2$$
$$\text{for all } k \in K, \text{ for all } f \in L^1(G)\}$$

A_K , ϵ is weak $*$ closed.

As the intersections of finitely many sets $A_{K,\epsilon}$ is non empty the compactness of ${}^1\mathbf{S}_I^N = \mathbf{S}_I^N \cap \{B \in \mathbf{S} \mid | B | \leqslant 1\}$ in the weak $*$ topology implies:

$$\bigcap_{\substack{K \text{compact} \\ \epsilon > 0}} A_{K,\epsilon} \neq \varnothing .$$

For a $\tilde{B}$ in this intersection we have: $\tilde{B}(f,f) = {}_h\tilde{B}(f,f)$ for all $f \in L^1(G)$, for all $h \in H$.

This implies: $\mathbf{S}_I^H \neq (0)$. Q.E.D.

A topological group G is said to be solvable, if it contains a finite chain of closed subgroups

$$G = G_0 \supset G_1 \cdots \supset G_{N+1} = \{e\}$$

such that G_{n+1} is normal in G_n and $G_n/_{G_{n+1}}$ is abelian.

$$(0 \leqslant n \leqslant N).$$

We have proved:

Discrete exponentially bounded groups and discrete solvable groups have the Wiener property.

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